Quantum mechanics II, Chapter 3 : Reduced and mixed quantum states

TA : Slimane Thabet, Sofia Brizigotti, Alba Miren Taddei, Reyhaneh Aghaei Saem, Mehrad Sahebi, Ricard Puig, Sacha Lerch

Problem 1 : Bloch sphere for pure or mixed states of a two-level system

In the lecture you have started to talk about density matrices. This exercise serves as a first introduction to the topic, connecting to the already known concept of the Bloch sphere.

1. *Derivation of Bloch vector from generic pure state.* Any pure one-qubit quantum state can be written as ket

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \quad \theta \in [0, \pi),\ \phi \in [0, 2\pi)$$

or as the density matrix,

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}\,\right)$$

Find an expression for $\mathbf{r}$ in terms of θ and ϕ. What does the vector $\mathbf{r}$ denote ?

We just have to expand the LHS and the RHS separately and then equate the matrix entries in a pairwise manner. Starting from the LHS, we have

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}, \tag{1}$$

while for the RHS we have

$$\frac{1}{2}(\mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{r}) = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}. \tag{2}$$

The diagonal entries yield the constraints $\cos^2\theta/2 = (1 + r_z)/2$ and $\sin^2\theta/2 = (1 - r_z)/2$, which can be subtracted to obtain

$$r_z = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta. \tag{3}$$

One can proceed similarly with the off-diagonal entries and find that by adding and subtracting the two associated equations, respectively, we get

$$r_x = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(e^{-i\phi} + e^{i\phi}) = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\phi = \sin\theta\cos\phi, \tag{4}$$

$$r_y = \sin\theta\sin\phi. \tag{5}$$

So we recover the Bloch vector

$$\boldsymbol{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{6}$$

which represents a unit vector in spherical coordinates in $\mathbb{R}^3$ (which is restricted to the boundary of the unit sphere).

2. *Derivation of Bloch vector from properties of density operators.* Define the set of density matrices with the following 3 conditions :
   — The density matrix is Hermitian : $\hat{\rho}^\dagger = \hat{\rho}$
   — It has trace 1 : $\mathrm{Tr}\hat{\rho} = 1$
   — It is positive or null : $\langle\Psi|\hat{\rho}|\Psi\rangle \geq 0, \quad \forall\Psi$

   Show that any density matrix $\hat{\rho}$ of the 2 level system can be written

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}), \tag{7}$$

   where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Argue that $\boldsymbol{r}$ is a real vector of 3D space and $|\boldsymbol{r}| \leq 1$.

We know that density matrices are Hermitian. We also already showed in Problem Set 1 that any Hermitian operator ρ can be written as a linear combination of the Pauli matrices (including the identity operator), that is,

$$\rho = a\mathbb{1} + b\sigma_x + c\sigma_y + d\sigma_z = a\mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{r}', \tag{8}$$

where $a, b, c, d \in \mathbb{R}$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\boldsymbol{r}' = (b, c, d)$. Density matrices must additionally satisfy $\mathrm{Tr}\rho = 1$ and positivity, which yield the following contraints :

$$\mathrm{Tr}\rho = \mathrm{Tr}[a\mathbb{1}] = 2a = 1 \implies a = 1/2 \tag{9}$$

$$\tag{10}$$

And eigenvalues have to be larger than 0. From (8) we have :

$$\rho = \begin{bmatrix} 1/2 + d & b - ic \\ b + ic & 1/2 - d \end{bmatrix} \tag{11}$$

$$\det[\lambda\mathbb{1} - \rho] = (\lambda - 1/2 - d)(\lambda - 1/2 + d) - b^2 - c^2 = 0 \tag{12}$$

$$= \lambda^2 - \lambda - |\mathbf{r}'|^2 + 1/4 \tag{13}$$

And therefore :

$$\lambda = 1/2 \pm |\mathbf{r}'| \geq 0 \tag{14}$$

$$|\mathbf{r}'| \leq 1/2 \tag{15}$$

$$|\mathbf{r}'| \geq -1/2 \tag{16}$$

All together we have $\rho = a\mathbb{1} + \mathbf{r}' \cdot \sigma = a\left(\mathbb{1} + \mathbf{r}'/a \cdot \sigma\right) = \frac{1}{2}\left(\mathbb{1} + \mathbf{r} \cdot \sigma\right)$ with $\mathbf{r} = 2\mathbf{r}'$ and $|\mathbf{r}| \leq 1$, as required.

Alternatively to the positivity one can also use $\mathrm{Tr}\{\rho^2\} \leq 1$, yielding :

$$\mathrm{Tr}\rho^2 = \mathrm{Tr}[a^2\mathbb{1} + 2a\boldsymbol{\sigma} \cdot \boldsymbol{r}' + (\boldsymbol{\sigma} \cdot \boldsymbol{r}')^2] = 2(a^2 + |\boldsymbol{r}'|^2), \tag{17}$$

where we used the fact that Pauli matrices are traceless, the linearity of the trace operation, and finally the rewriting $(\boldsymbol{\sigma} \cdot \boldsymbol{r}')^2 = \boldsymbol{r}' \cdot \boldsymbol{r}'\mathbb{1} + i(\boldsymbol{r}' \times \boldsymbol{r}')_k\sigma_k = |\boldsymbol{r}'|^2\mathbb{1}$ (see Problem Set 1). Putting everything together, we find

$$\frac{1}{2} + 2|\boldsymbol{r}'|^2 \leq 1 \quad \implies \quad |\boldsymbol{r}'|^2 \leq \frac{1}{4} \quad \implies \quad |\boldsymbol{r}| \leq 1, \tag{18}$$

provided $\boldsymbol{r}$ is chosen such that $\boldsymbol{r} = 2\boldsymbol{r}'$.

Notice that in the general case of mixed states, the Bloch vector can occupy the entire unit sphere – not just its boundary as for pure states.

3. Show that the state is pure iff $||\boldsymbol{r}|| = 1$. Explain why $\mathrm{Tr}[\rho^2]$ is a measure of the 'purity' of a quantum state.

Since it is a iff (if and only if statement), we must prove *both* directions of the condition. If we first assume that the state is pure, we have already seen in the first question, what the representation of the state is and that $|r| = 1$. Conversely, if we have $|r| = 1$ we know that we can represent any point on the surface of the unit sphere. Since after exercise 1, any point on the surface of the unit sphere is associated to a pure state, $|r| = 1$ implies that the state is pure.

Alternatively, for pure states we also have $\rho^2 = \rho$ (by definition in the first exercise), and therefore $\mathrm{Tr}\rho^2 = 1$, which means that the following condition must hold

$$\mathrm{Tr}\rho^2 = \mathrm{Tr}\left[\left(\frac{1}{2}(\mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{r})\right)^2\right] = \mathrm{Tr}\left[\frac{1}{4}\mathbb{1} + \frac{1}{4}|\boldsymbol{r}|^2\mathbb{1}\right] = \frac{1}{2} + \frac{1}{2}|\boldsymbol{r}|^2 = 1, \tag{19}$$

which is only satisfied when $|\boldsymbol{r}| = 1$. If on the other hand we assume that $|\boldsymbol{r}| = 1$, then from the formula above for $\mathrm{Tr}\rho^2$ we can directly conclude that necessarily the state is pure, that is, $\mathrm{Tr}\rho^2 = 1$.

$\mathrm{Tr}\rho^2$ is a good measure for purity because only pure states have $\mathrm{Tr}\rho^2 = 1$. For a fully mixed two-level state we have $\rho = \sum_i \frac{1}{2} |i\rangle \langle i|$ and therefore $\rho^2 = \sum_{i,j} \frac{1}{4} \sum_i |i\rangle \langle i|$ and $\mathrm{Tr}\rho^2 = \frac{1}{2}$.
In general for a Hilbert space of dimension d the maximally mixed state has $\mathrm{Tr}\rho^2 = 1/d$.

4. Sketch on the Bloch sphere the states :
   a) $|0\rangle$
   b) $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   c) $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
   d) $\frac{1}{\sqrt{3}}(|0\rangle - i\sqrt{2}\,|1\rangle)$.
   e) $\frac{1}{2}\mathbb{1}$
   f) $\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$

We employ the following two tricks to draw the one-qubit mixed states on the Bloch sphere :

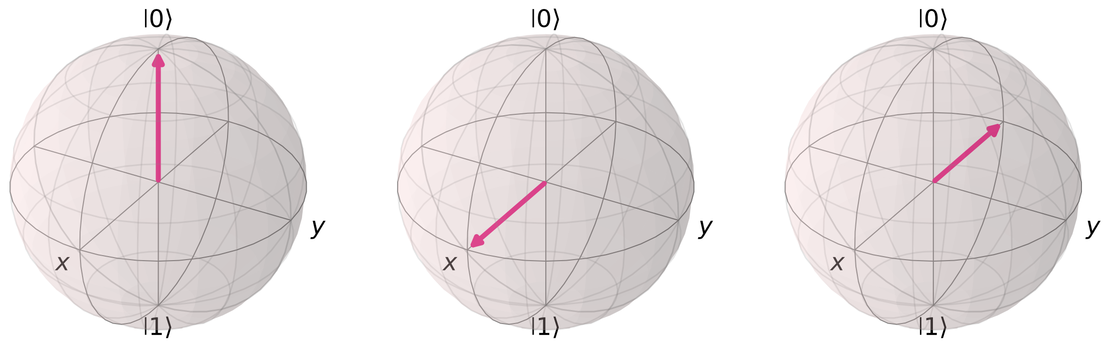
- A density matrix $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r})$ occupies the position $\boldsymbol{r}$ on the Bloch sphere (which we traditionally draw as a vector from the origin).
- A density matrix $\hat{\rho} = a\,\hat{\rho}_1 + b\,\hat{\rho}_2$, with respective vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, has a Bloch vector $\boldsymbol{r} = a\,\boldsymbol{r}_1 + b\,\boldsymbol{r}_2$.

In Fig. 1 we sketch the states on the Bloch sphere.

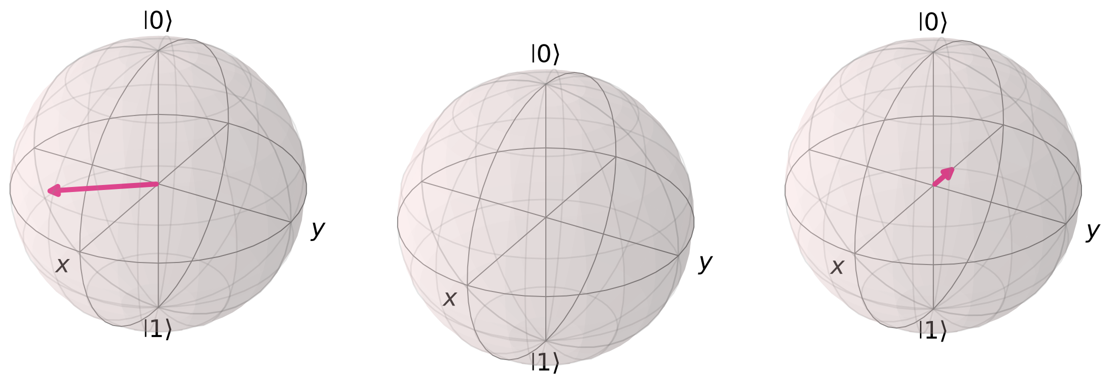
5. Give a geometric argument to show that $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. (Is this surprising ?)

   Disclaimer : think about the meaning of this state. How would you represent $|0\rangle \langle 0|$ on the Bloch sphere ? And $|1\rangle \langle 1|$? Now if the space of qubits is a convex space, what is the point in the Bloch sphere that represents the combination of the previous one ?

   Now make the same reasoning for $|+\rangle \langle +|$, $|-\rangle \langle -|$.



(a) $|0\rangle$, $\theta = 0$ and $\phi = 0$. (b) $|+\rangle$, $\theta = \pi/2$ and $\phi = 0$. (c) $|-\rangle$, $\theta = \pi/2$ and $\phi = \pi$.



(d) $(|0\rangle - i\sqrt{2}|1\rangle)/\sqrt{3}$, $\theta = 2\arccos(1/\sqrt{3})$ and $\phi = 3\pi/2$. (e) $\mathbb{1}/2$, $\boldsymbol{r} = (0, 0, 0)$. (f) $|+\rangle\langle+|/3 + 2|-\rangle\langle-|/3$, $\boldsymbol{r} = (-1/3, 0, 0)$.

FIGURE 1 – Bloch sphere representation on quantum states.

Notice that we have

$$\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}, \tag{20}$$

which, as shown above, corresponds to the origin of the Bloch sphere, i.e. $\boldsymbol{r} = (0, 0, 0)$. This is expected since $\{|+\rangle, |-\rangle\}$ and $\{|0\rangle, |1\rangle\}$ both form a basis, and as seen from the Bloch sphere representation, they cancel out.
Also by definition of the ortho-normal basis :

$$\mathbb{1} = \sum_i |i\rangle \langle i| \tag{21}$$

$$U\mathbb{1}U^\dagger = \sum_i U |i\rangle \langle i| U^\dagger \tag{22}$$

$$\mathbb{1} = \sum_\alpha |\alpha\rangle \langle\alpha| \tag{23}$$

where U is the matrix describing a change of basis from span$\{|i\rangle\}$ to span$\{|\alpha\rangle\}$ (In this case from Z-basis to X-basis). So we have

$$\sum_i |i\rangle \langle i| = \mathbb{1} = \sum_\alpha |\alpha\rangle \langle\alpha|\,, \tag{24}$$

as desired.

Problem 2 : No signalling

Use your new-found understanding of reduced states to justify the no signalling principle (i.e. to argue why it is not possible to use entanglement to signal faster than light).

Intuitively

We can immediately intuit why signalling is impossible using our understanding of the reduced density matrix, which we know it is a description of a partition totally independent of the other partition(s). No matter what is performed upon the other partitions, the reduced density matrix is unchanged. Because the statistics of local measurements are informed entirely by expected values of operators upon the reduced density matrix, they are also independent of operations on other partitions. Ergo, signalling is impossible.

Two qubits

We can also prove this using density matrices, which we will find to be a much more pleasant process than our previous proofs using only statevectors. For simplicity, let's first consider Alice and Bob each have a qubit in a general, two-qubit pure state. This includes all possible entangled states. Alice proposes to perform an arbitrary measurement upon her qubit, collapsing it into either outcome state $|\lambda_1\rangle$ or $|\lambda_2\rangle$. We can use these states as a basis to express the general, pre-measurement state as

$$|\psi\rangle = \alpha\,|\lambda_1\rangle\,|\phi_1\rangle + \beta\,|\lambda_2\rangle\,|\phi_2\rangle\,, \tag{25}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\phi_i\rangle$ is Bob's corresponding state. The Born rule permits us to interpret $|\alpha|^2$ as the probability of Alice measuring λ_1, and $|\beta|^2 = 1 - |\alpha|^2$ as the probability she measures λ_2. Let's notate these as $p_1 = |\alpha|^2$ and $p_2 = |\beta|^2$.

Before we consider actually performing any measurement, let us first compute the reduced density matrix ρ_B of Bob's qubit via the partial trace of this arbitrary state. We trace out Alice's qubit, choosing her outcome states $|\lambda_i\rangle$ as the enumerated basis.

$$\rho_B = \text{Tr}_A\Big(|\psi\rangle\langle\psi|\Big) = \sum_i \Big(\langle\lambda_i| \otimes \hat{\mathbb{1}}\Big)\,|\psi\rangle\langle\psi|\,\Big(|\lambda_i\rangle \otimes \hat{\mathbb{1}}\Big). \tag{26}$$

We substitute in our general pure state

$$|\psi\rangle\langle\psi| = \Big(\alpha\,|\lambda_1\rangle\,|\phi_1\rangle + \beta\,|\lambda_2\rangle\,|\phi_2\rangle\Big)\Big(\alpha^*\,\langle\lambda_1|\,\langle\phi_1| + \beta^*\,\langle\lambda_2|\,\langle\phi_2|\Big) \tag{27}$$

$$= |\alpha|^2\,|\lambda_1,\phi_1\rangle\langle\lambda_1,\phi_1| + \alpha\beta^*\,|\lambda_1,\phi_1\rangle\langle\lambda_2,\phi_2| + \beta\alpha^*\,|\lambda_2,\phi_2\rangle\langle\lambda_1,\phi_1| + |\beta|^2\,|\lambda_2,\phi_2\rangle\langle\lambda_2,\phi_2| \tag{28}$$

although spare ourselves the nuisance of handling every projector by appreciating that $\langle\lambda_1|\lambda_2\rangle = 0$, so that the partial trace simplifies to

$$\rho_B = |\alpha|^2\,|\phi_1\rangle\langle\phi_1| + |\beta|^2\,|\phi_2\rangle\langle\phi_2| \tag{29}$$

$$= p_1\,|\phi_1\rangle\langle\phi_1| + p_2\,|\phi_2\rangle\langle\phi_2|\,. \tag{30}$$

Let us now consider that Alice *does* perform her measurement. The shared state collapses to either

$$|\psi\rangle \to \begin{cases} |\lambda_1\rangle\,|\phi_1\rangle\,, & \text{with probability } p_1, \\ |\lambda_2\rangle\,|\phi_2\rangle\,, & \text{with probability } p_2. \end{cases} \tag{31}$$

The post-measurement system can be in one of multiple states as per the specified probabilities. We can describe such a state, encoding the classical randomness (i.e. the outcome state) using a density matrix ! The post-measurement mixed state is simply written down as :

$$\rho' = p_1 \left(\rho_{A=\lambda_1}\right) + p_2 \left(\rho_{A=\lambda_2}\right) \tag{32}$$
$$= p_1 \, |\lambda_1, \phi_1\rangle \langle \lambda_1, \phi_1| + p_2 \, |\lambda_2, \phi_2\rangle \langle \lambda_2, \phi_2| \, . \tag{33}$$

Let us now again trace out Alice's qubit to obtain the reduced density matrix of Bob's qubit.

$$\rho'_B = \mathrm{Tr}_A(\rho) = \sum_i \Big(\langle \lambda_i| \otimes \hat{\mathbb{1}} \Big) \rho \Big(|\lambda_i\rangle \otimes \hat{\mathbb{1}} \Big) \tag{34}$$
$$= p_1 \, |\phi_1\rangle \langle \phi_1| + p_2 \, |\phi_2\rangle \langle \phi_2| \, . \tag{35}$$

Lo and behold, this is precisely the expression we found for Bob's state when Alice did *not* perform a prior measurement. We have ergo proven that Alice's measurement has no affect on Bob's state, nor the statistics of his subsequent measurements. Alice *cannot* communicate to Bob via her measurement using two shared qubits, entangled or otherwise.

Any number of any-level systems

If we want to be really rigorous, we should generalise our proof to permit Alice and Bob to each have *any* dimension subspaces (e.g. many *qutrits*, or their own continuously parameterised systems !). For illustration, let's now do this using a different logic than used above, which will make use of some properties of projectors and traces you have not yet seen ! We permit Alice and Bob to each have one partition of any quantum state $|\psi\rangle$. We'll notate operators upon their respective partitions as $\hat{A} \otimes \hat{\mathbb{1}}$ and $\hat{\mathbb{1}} \otimes \hat{B}$ respectively.

Let $\{\hat{\Pi}_i\}$ be projectors corresponding to Alice's possible outcomes when performing some measurement on her partition. Given Alice is no longer measuring one qubit, there could be many more than two such projectors. The possible outcome states can be expressed in terms of their projectors as :

$$|\psi\rangle \to \Big\{ \, |\psi_i\rangle = \frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) \, |\psi\rangle \, : \; i \Big\}, \tag{36}$$

where we have renormalised the post-projector states via the probabilities of their corresponding measurement outcomes, $p_i = \langle \psi| \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) |\psi\rangle$. The output state after Alice's measurement can be expressed as a single mixed state :

$$\rho = \sum_i p_i \, |\psi_i\rangle \langle \psi_i| = \sum_i p_i \left(\frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) \right) |\psi\rangle \langle \psi| \left(\frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) \right)^\dagger = \sum_i \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) |\psi\rangle \langle \psi| \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big), \tag{37}$$

where we leveraged that projectors are self-adjoint, i.e. $\hat{\Pi}_i = \hat{\Pi}_i^\dagger$. The reduced density matrix of Bob's partition after Alice's measurement is

$$\rho_B = \mathrm{Tr}_A(\rho) = \mathrm{Tr}_A \left(\sum_i \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) |\psi\rangle \langle \psi| \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\Big) \right). \tag{38}$$

Happily, we will do need even need to evaluate this partial trace! We can instead simplify it using some of its properties, such as linearity :

$$\rho_B = \sum_i \text{Tr}_A\left(\left(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\left(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\right)\right) \tag{39}$$

We will next shuffle around some operators. Beware that unlike the trace which is cyclic, i.e. $\text{Tr}\Big(\hat{L}_1\hat{L}_2\hat{L}_3\Big) = \text{Tr}\Big(\hat{L}_3\hat{L}_1\hat{L}_2\Big) = \text{Tr}\Big(\hat{L}_2\hat{L}_3\hat{L}_1\Big)$, the patrial trace is only cyclic with respect to operators of the traced subspace. This is easy to demonstrate :

$$\text{Tr}_{\text{left}}\left((\hat{L}_1 \otimes \hat{R}_1)(\hat{L}_2 \otimes \hat{R}_2)(\hat{L}_3 \otimes \hat{R}_3)\right) = \text{Tr}_{\text{left}}\left((\hat{L}_1\hat{L}_2\hat{L}_3) \otimes (\hat{R}_1\hat{R}_2\hat{R}_3)\right) \tag{40}$$

$$= \text{Tr}\left(\hat{L}_1\hat{L}_2\hat{L}_3\right)(\hat{R}_1\hat{R}_2\hat{R}_3) = \text{Tr}\left(\hat{L}_3\hat{L}_1\hat{L}_2\right)(\hat{R}_1\hat{R}_2\hat{R}_3) = \text{Tr}\left(\hat{L}_2\hat{L}_3\hat{L}_1\right)(\hat{R}_1\hat{R}_2\hat{R}_3) \tag{41}$$

$$= \text{Tr}_{\text{left}}\left((\hat{L}_3 \otimes \hat{R}_1)(\hat{L}_1 \otimes \hat{R}_2)(\hat{L}_2 \otimes \hat{R}_3)\right) = \text{Tr}_{\text{left}}\left((\hat{L}_2 \otimes \hat{R}_1)(\hat{L}_3 \otimes \hat{R}_2)(\hat{L}_1 \otimes \hat{R}_3)\right) \tag{42}$$

and holds true even when some operators are not separable (like $|\psi\rangle\langle\psi|$ in our case), which we could show by expressing them as a weighted sum in a separable basis and expanding via linearity. By cyclity, Bob's state becomes

$$\rho_B = \sum_i \text{Tr}_A\left(\left((\hat{\Pi}_i\hat{\Pi}_i) \otimes \hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right)\right) \tag{43}$$

$$= \sum_i \text{Tr}_A\left(\left(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\right), \tag{44}$$

via idempotency of $\hat{\Pi}_i$ (i.e. $\hat{\Pi}_i\hat{\Pi}_i = \hat{\Pi}_i$). Let's now move the sum around the only terms affected by it, utilising linearity of both the partial trace and the tensor product, to express

$$\rho_B = \text{Tr}_A\left(\left(\left(\sum_i \hat{\Pi}_i\right) \otimes \hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\right). \tag{45}$$

Finally, we recognise that the sum of projectors of all orthonormal outcome states is the identity operator ;

$$\sum_i \hat{\Pi}_i = \mathbb{1} \tag{46}$$

To appreciate this, think about applying it to any particular state, when expressing that state in this perfectly valid basis. Our algebra above has concluded that the reduced density matrix describing Bob's qubit after Alice's measurement is :

$$\rho_B = \text{Tr}_A\left(|\psi\rangle\langle\psi|\right). \tag{47}$$

We immediately recognise this is identical to Bob's reduced density matrix when Alice does *not* perform any prior measurement. Alice cannot signal to Bob via her measurement basis no matter *what* quantum state they share!

Problem 3 : Density operator, partial trace, information and measure

Alice and Bob share state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{48}$$

1. What is the density matrix of the system $\hat{\rho}$ with 2 qubits ?
2. Verify that it is a pure state by calculating $\mathrm{Tr}(\hat{\rho}^2)$.

Note $\hat{\rho}_B = \mathrm{Tr}_A\,\hat{\rho}$ the density matrix obtained by partial trace on Alice's qubit. This matrix is an operator on the Hilbert space of the second qubit (Bob's), and reflects the information available to Bob.

3. Calculate $\hat{\rho}_B$ and link that result to the probability Bob has to get outcome 0 or 1 when he measures his qubit in the computational basis (We will write $\hat{O}$ the corresponding observable). Also verify that we have $\langle \hat{O} \rangle = \mathrm{Tr}[\hat{\rho}(\mathbb{1} \otimes \hat{O})] = \mathrm{Tr}(\hat{\rho}_B \hat{O})$.
4. Does the matrix ρ_B describe a pure state of the second qubit ? Justify by calculating $\mathrm{Tr}(\hat{\rho}_B^2)$. What about if the 2 qubit state $|\psi\rangle$ being separable in the form $|\psi_A\rangle \otimes |\psi_B\rangle$? We sometimes say that statistical mixtures of the state of a system is the fruit of entanglement of this system with its environment ; how can we interpret this in the light of the previous results ?

We admit that when the measurement of an observable $\hat{M}$ on the system gives the result m, then the density matrix ($\hat{\rho}$ before measurement) reads

$$\hat{\rho}' = \frac{\hat{P}_m \hat{\rho} \hat{P}_m^\dagger}{\mathrm{Tr}(\hat{P}_m^\dagger \hat{P}_m \hat{\rho})}, \tag{49}$$

where $\hat{P}_m$ is the projector on the subspace relative to m.

5. What is the state with 2 qubits $|\psi'\rangle$ obtained when Alice measure her qubit in the computational basis and finds 0 ? Compare $|\psi'\rangle \langle\psi'|$ and $\hat{\rho}'$.
6. When Alice measures her qubit on state $|\psi\rangle$ and finds 0, what is the density matrix ? $\hat{\rho}'_B$? Comment.

1. The density matrix of the system is

$$\rho = |\psi\rangle \langle\psi| = \frac{1}{2}|01\rangle\langle 01| - \frac{1}{2}|01\rangle\langle 10| - \frac{1}{2}|10\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10|, \tag{50}$$

and its matrix form in the computational basis reads

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{51}$$

2. We have $\hat{\rho}^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \hat{\rho}$, which we could also verify by squaring (50) and (51). This implies $\mathrm{Tr}(\hat{\rho}^2) = 1$ and characterises a pure state (which we knew by construction).
3. The reduced density matrix describing Bob's qubit is given by the partial trace, tracing over Alice's basis states.

$$\rho_B = \mathrm{Tr}_A(\rho) = \sum_{|\phi\rangle \in \{|0\rangle, |1\rangle\}} \left(\langle\phi| \otimes \hat{\mathbb{1}}\right) \rho \left(|\phi\rangle \otimes \hat{\mathbb{1}}\right) \tag{52}$$

You can expand and evaluate this in the usual approach, although it is perhaps clearer to explicitly invoke that the matrix element $(\rho_B)_{ij}$ of the reduced state ρ_B is given by

$$\langle i|\rho_B|j\rangle = \sum_x \langle x|\langle i|\,\rho\,|x\rangle|j\rangle. \tag{53}$$

We can compute each of the 4 matrix elements separately. Using Eq. (50), the first element is

$$\langle 0|\rho_B|0\rangle = \sum_x \langle x0|\rho|x0\rangle = \langle 00|\rho|00\rangle + \langle 10|\rho|10\rangle = 0 + \frac{1}{2} = \frac{1}{2}. \tag{54}$$

One can proceed similarly to get the other 3 matrix elements. The reduced density matrix of Bob in matrix form is then

$$\rho_B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tag{55}$$

which corresponds to a statistical mixture of $|0\rangle$ and $|1\rangle$, with equal probability 1/2.

If $O \equiv |1\rangle\langle 1|$, we now want to show that $\langle O\rangle = \mathrm{Tr}[\rho(\mathbb{1}\otimes O)] = \mathrm{Tr}(\rho_B O)$. In matrix form, the observable O is given by

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{56}$$

Using matrix notation we find the result of interest

$$\mathrm{Tr}[\rho(\mathbb{1}\otimes O)] = \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}, \tag{57}$$

$$\mathrm{Tr}(\rho_B O) = \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}. \tag{58}$$

4. The density matrix ρ_B describes a mixed state since $\mathrm{Tr}(\rho_B^2) = \mathrm{Tr}(\mathbb{1}/4) = 1/2 < 1$. If $|\psi\rangle$ is separable in the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then the density matrix becomes

$$\rho = |\psi_A\psi_B\rangle\langle\psi_A\psi_B| = \rho_1 \otimes \rho_2, \tag{59}$$

with

$$\rho_1 = |\psi_A\rangle\langle\psi_A|, \tag{60}$$
$$\rho_2 = |\psi_B\rangle\langle\psi_B|, \tag{61}$$

and the density matrix of the second qubit, that is

$$\hat{\rho}_B = \mathrm{Tr}_A(\hat{\rho}_1 \otimes \hat{\rho}_2) = (\mathrm{Tr}\hat{\rho}_1)\hat{\rho}_2 = \hat{\rho}_2 = |\psi_B\rangle\langle\psi_B|, \tag{62}$$

is indeed a pure state. We have thus illustrated by an example, the fact that if the total system is a pure state, a subsystem appears mixed iff it is entangled with the rest of the system.

5. If Alice measures her qubit and finds 0, it means the 2 qubits state obtained is $|\psi'\rangle = |01\rangle$, in which case

$$|\psi'\rangle\langle\psi'| = |01\rangle\langle 01|. \tag{63}$$

Now to compute ρ', we need the projector on the subspace 0, that is,

$$P_0 = |0\rangle\langle 0| \otimes \mathbb{1} = P_0^\dagger. \tag{64}$$

The computation is then straightforward

$$P_0\rho = \frac{1}{2}\left|01\right\rangle\left\langle 01\right| - \frac{1}{2}\left|01\right\rangle\left\langle 10\right|, \tag{65}$$

$$\hat{P}_0\rho P_0^\dagger = \frac{1}{2}\left|01\right\rangle\left\langle 01\right|, \tag{66}$$

$$\mathrm{Tr}(P_0^\dagger P_0\rho) = \mathrm{Tr}(P_0\rho P_0^\dagger) = \frac{1}{2}, \tag{67}$$

and the new density operator is

$$\rho' = \frac{P_0\rho P_0^\dagger}{\mathrm{Tr}(P_0^\dagger P_0\rho)} = \left|01\right\rangle\left\langle 01\right| = \left|\psi'\right\rangle\left\langle\psi'\right|, \tag{68}$$

6. Again, if Alice measures her qubit and finds 0, it means the 2 qubits state obtained is $|\psi'\rangle = |01\rangle = |0\rangle_A \otimes |1\rangle_B$. That is, it is separable (or, equivalently, it is a "product state"). As a result, according to the discussion above about the density matrix of separable states, we can directly write down Bob's reduced density matrix as $\rho'_B = |1\rangle\langle 1|$.